Let $|G| = p^n m$ with $(p, m) = 1$. Let n_p be the number of Sylow p-subgp of G. Then

$$n_p \mid m \quad \text{and} \quad n_p \equiv 1 \mod p$$

Simple applications (finite gps of certain order are not simple)

① $|G| = 15$. Then $n_5 \mid 3$ and $n_5 \equiv 1 \mod 5$.

So $n_5 = 1$. Thus the Sylow 5-subgp P of G is normal.

Then $\{1\} < P < G$. So G not simple

② $|G| = 20$. Then $n_5 \mid 4$ and $n_5 \equiv 1 \mod 5$.

So $n_5 = 1$. Thus the Sylow 5-subgp P of G is normal.

$|G/P| = 4$. So G/P is abelian. Thus G not simple

Lem. If G has k subgp of order p, then G has $k(p-1)$ elt of order p.

Pf. $\{\text{elt of order } p\} \longrightarrow \{\text{subgp of order } p\}$

$$g \longmapsto \langle g \rangle$$

This is a $(p-1):1$ map as each subgp contains $(p-1)$ of such elt. □

③ $|G| = 12$. Then $n_2 = 1$ or 3, $n_3 = 1$ or 4.

If $n_3 = 1$. Then G not simple

If $n_3 = 4$, then G has $4 \cdot 2 = 8$ elt of order 3.
The remaining $12 - 8 = 4$ elt must form a Sylow 2-subgp.
So $n_2 = 1$. Again G not simple

④ $|G| = 30$. Then $n_3 = 1$ or 10, $n_5 = 1$ or 6.
If $n_3 = 1$ or $n_5 = 1$, then G not simple
If $n_3 = 10$ & $n_5 = 6$, then G has 20 elt of order 3
24 elt of order 5.
Contradiction.

Lem. If $H, K < G$, then $|HK| = |H||K| / |H \cap K|$.
Pf. Consider the map $\pi : H \times K \longrightarrow HK$.
Then each fiber is in bijection with $H \cap K$ $\square$.

⑤ $|G| = 48$. Then $n_2 = 1$ or 3.
If $n_2 = 1$, then G is not simple
If $n_2 = 3$, let H, K be Sylow 2-subgp (of order 16)
Then $|HK| \le 48 \Rightarrow |H \cap K| = 8$. So $H \cap K \triangleleft H, K$.
Now $N_G(H \cap K) \supseteq H, K$. So $N_G(H \cap K) = G$. Then $H \cap K \trianglelefteq G$.
G not simple

⑥ $|G|=36$. Then $n_3 = 1$ or 4.

If $n_3 = 1$, then G is not simple

If $n_3 = 4$, let H, K be Sylow subgp (of order 9)

Then $|HK| \leq 36 \Rightarrow |H \cap K| = 3$.

By 1st Sylow Th, $H \cap K \trianglelefteq H, K$.

So $|N_G(H \cap K)| > 9$. Thus

(i) $|N_G(H \cap K)| = 36$. So $H \cap K \trianglelefteq G$. G is not simple

(ii) $|N_G(H \cap K)| = 18$. So $N_G(H \cap K) \trianglelefteq G$ (since index $= 2$)

So G is not simple

Application (classification of finite gp of given order)

Lem. Let $p \neq q$ be prime factors of $|G|$. If $n_p = n_q = 1$, then elt in the Sylow p-subgp commutes with elt in the Sylow q-subgp

Pf. Let P, Q be the Sylow p-subgp and Sylow q-subgp.

Then $P, Q \trianglelefteq G$. Also $P \cap Q = \{1\}$.

Now $\forall a \in P, b \in Q$, $aba^{-1}b^{-1} \in P \cap Q = \{1\}$. So

$ab = ba$ □

Prop. All Sylow subgp of G are normal iff G iso to the direct product of its Sylow subgp

Pf. $\Leftarrow$ by the def of direct product.

$\Rightarrow$ Let $P_1, \dots, P_l$ be Sylow gp of G.

Consider the map $\pi: P_1 \times \cdots \times P_l \to G$

This map is injective, hence bijective as both sides have the same order.

Also a gp hom (by the above lem) $\square$

Th. Let $|G| = pq$ with $q > p$. Then G is not simple.

If moreover $q \not\equiv 1 \mod p$, then $G \cong \mathbb{Z}_{pq}$.

Pf. $n_q = 1$. So $Q \trianglelefteq G$.

If $q \not\equiv 1 \mod p$, then $n_p = 1$. So $G \cong P \times Q \cong \mathbb{Z}_p \times \mathbb{Z}_q \cong \mathbb{Z}_{pq}$. $\square$

Th. Let $|G| = p^2 q$. Then G is not simple.

If moreover, $q > p$ and $q \not\equiv 1 \mod p$. then G is abelian

Pf. If $p > q$, then $n_p = 1$. So G is not simple

If $q > p$. then $n_q \mid p^2$ and $n_q \equiv 1 \mod q$

Thus $n_q = 1$ (So G not simple) or $n_q = p^2$.

If $n_q = p^2$. Then $q \mid (p^2 - 1) \Rightarrow p = 2, q = 3$. So $|G| = 12$

We have proved that G is not simple in this case.

If $q \not\equiv 1 \mod p$, then $n_p = n_q = 1$.

So $G \cong P \times Q$. Here $|P| = p^2$. So P is abelian

$|Q| = q$. So Q is cyclic $\square$.

Th. Let $|G| = p^2 q^2$. Then G is not simple.

Pf. Suppose $q > p$. Then $n_q \mid p^2$ and $n_q \equiv 1 \mod p$.

So either $n_q = 1$ (So G is not simple)

or $p = 2, q = 3$. So $|G| = 36$.

We have shown that a gp of order 36 is not simple $\square$